# Non-invertible and chiral symmetries I

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Croucher Summer Course, Emergent Symmetry

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# Chiral and non-invertible symmetry

- Chiral symmetries, roughly speaking, are symmetries that act differently on the left and right degrees of freedom.

- Chiral symmetries play an important role in particle physics, in the context of the pion Lagrangian, confinement, and more.

- Many non-invertible symmetries are closely related to the underlying chiral symmetries.

- I will discuss this connection in a couple of examples.

# Symmetries and anomalies between UV and IR

# Emergent Symmetry

- Consider a UV system whose long-distance behavior is described by an IR system.
- Let $G_{UV}$ and $G_{IR}$ be the faithfully-acting global symmetries in the UV and IR, resp. They might each have their own 't Hooft anomalies.
- These two symmetries can be different. In any case, there is a homomorphism

$$\rho: G_{UV} \to G_{IR}$$

- $\rho$ need **not** be onto. Some elements in $G_{IR}$ may not lie in the image $\rho(G_{UV})$. This subgroup of $G_{IR}$ is the emergent symmetry.

# Emanant Symmetry

$$\rho: G_{UV} \to G_{IR}$$

- There is a more interesting possibility:

$$G_{IR} = \rho(G_{UV}) \text{ but } G_{UV} \neq G_{IR}$$

- Some of such symmetries are called emanant symmetries [Cheng-Seiberg ’22].

- In this case, the IR symmetry doesn’t emerge out of nowhere; it emanates from some symmetry in the UV.

# ‘t Hooft Anomalies

$$\rho: G_{UV} \to G_{IR}$$

- There are many definitions of the ‘t Hooft anomaly of a global symmetry $G$.
- To name a few, the obstruction to gauging, projective algebra, boundary of an SPT phase, etc.
- In most (if not all) of these definitions, anomaly implies an obstruction to a trivially gapped phase, i.e., a unique ground state on any spatial manifold, with an order one gap to the excited states.
- Importantly, the ‘t Hooft anomalies $\omega_{UV}$ and $\omega_{IR}$ maybe different for $G_{UV}$ and $G_{IR}$.
- The order $\mathrm{ord}(\omega)$ of the anomaly is defined as the minimal integer such that $G$ is anomaly-free in $\mathrm{ord}(\omega)$ copies of the system.
- Universal inequality [LewSmith-Pace-SHS ‘26]: $\mathrm{ord}(\omega_{UV}) \leq \mathrm{ord}(\omega_{IR})$

# Quantum-mechanical example: a qutrit

- Consider a single qutrit whose Hilbert space is 3-dimensional [LewSmith-Pace-SHS '26].
- Let the Hamiltonian be

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{pmatrix}$$

- UV: 3-dim Hilbert space. IR: 2-dim ground space.
- It has a UV global symmetry $G_{UV}$ generated by

$$A = X \oplus 1,\ B = Z \oplus 1,\ C = (-1_{2\times 2}) \oplus 1$$

They obey $AB = CBA, A^2 = B^2 = C^2 = 1$, which forms a dihedral group $D_8$ of order 8.

- $G_{UV}$ is realized linearly. It is anomaly-free, $\mathrm{ord}(\omega_{UV}) = 1$.

# Quantum-mechanical example: a qutrit

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{pmatrix}$$

- UV: 3-dim Hilbert space. IR: 2-dim ground space.
- It has a UV global symmetry $G_{UV} = D_8$ generated by

$$A = X \oplus 1,\ B = Z \oplus 1,\ C = (-1_{2\times 2}) \oplus 1$$

- The operator $C$ acts as minus the identity operator in the IR.
- $G_{IR}$ is a quotient of $G_{UV}$ and is generated by $A, B$:

$$AB = -BA$$

- $G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2$ is realized projectively. It is anomalous, $\mathrm{ord}(\omega_{IR}) = 2$.

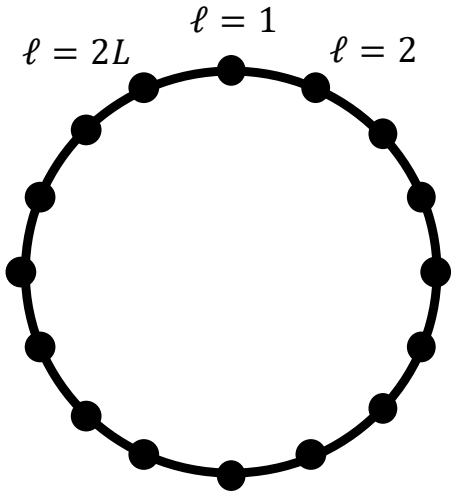
$G_{UV} = D_8$
(anomaly-free)

↓

$G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2$
(anomalous)

# Chiral symmetry and the Kitaev chain

# Kitaev chain




- 1+1d quantum lattice model of $2L$ sites labeled by $\ell = 1,2,\cdots,2L$. Periodic boundary condition.
- Each site has a single Majorana fermion $\chi_\ell = \chi_\ell^\dagger$.

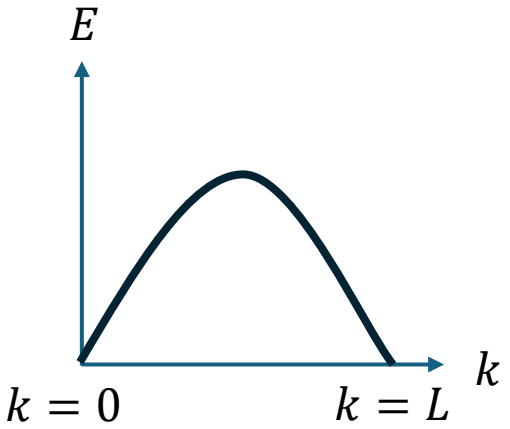
$$\{\chi_\ell, \chi_{\ell'}\} = 2\delta_{\ell\ell'}$$

- Kitaev chain Hamiltonian (UV) at the critical point:

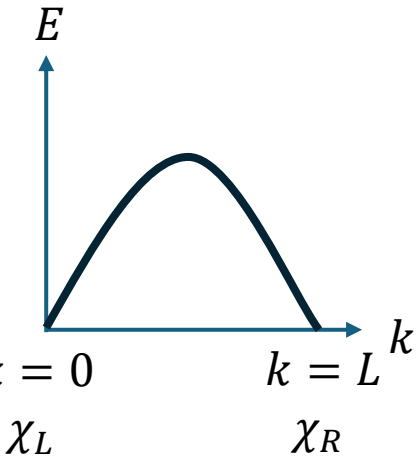
$$H = -i\sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$$

- In momentum space

$$H = 2\sum_{0<k<L} \sin\frac{2\pi k}{2L}\, \chi_{-k}\chi_k$$



# UV and IR symmetries




- UV Kitaev chain

$$H = -i \sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$$

- $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$

$$(-1)^F \chi_\ell (-1)^F = -\chi_\ell$$

$$T^{Maj} \chi_\ell \left(T^{Maj}\right)^{-1} = \chi_{\ell+1}$$

- IR Majorana fermion

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t + \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t - \partial_x)\chi_R$$

- $G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2^R$

$$(-1)^F \chi_L (-1)^F = -\chi_L,$$
$$(-1)^F \chi_R (-1)^F = -\chi_R$$

$$(-1)^{F_R} \chi_L (-1)^{F_R} = \chi_L,$$
$$(-1)^{F_R} \chi_R (-1)^{F_R} = -\chi_R$$

$G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2^R$ is a **quotient** of $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$

# Lattice chiral symmetry

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t + \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t - \partial_x)\chi_R$$

- Can the IR discrete chiral symmetry $\mathbb{Z}_2^R: \chi_R \to -\chi_R$ be realized as a $\mathbb{Z}_2$ group on this lattice? **NO!** [Jones-Metlitski '19, Zhang '23]
- Mass deformation

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t + \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t - \partial_x)\chi_R + im\chi_L\chi_R$$

- The discrete chiral symmetry $\mathbb{Z}_2^R: \chi_R \to -\chi_R$ exchanges $m > 0$ and $m < 0$, which correspond to two distinct invertible topological phases.

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t + \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t - \partial_x)\chi_R + im\chi_L\chi_R$$

# Finite-depth local unitary

- On the lattice, the mass deformation corresponds to:

$$H = -i\mu^{-1}\sum_j \chi_{2j}\chi_{2j+1} - i\mu\sum_j \chi_{2j-1}\chi_{2j}$$

- Trivial and nontrivial Kitaev chains; they **cannot** be related by a finite-depth local unitary (FDLU) [Huang-Chen '14].
- Therefore, the discrete chiral symmetry $\mathbb{Z}_2^R: \chi_R \to -\chi_R$ **cannot** be realized by a FDLU on this lattice.
- The index $\nu(U)$ for locality-preserving unitaries (a.k.a. QCA) $U$ satisfies

1. $\nu(U) = 0$ if $U$ is FDLU.
2. $\nu(U_1U_2) = \nu(U_1) + \nu(U_2)$

- It follows that $\mathbb{Z}_2^R: \chi_R \to -\chi_R$ **cannot** be realized by a unitary $U$ such that $U^2 = 1$. Q.E.D.

# Lattice chiral anomaly

- This is the simplest example of the broader no-go theorems: obstruction to realizing lattice chiral symmetries [Nielsen-Ninomiya, …]
- $G_{UV}$ must be different from the IR chiral symmetry group $G_{IR}$.
- Underlying reason: 't Hooft anomalies for the chiral symmetry.
- The anomaly of the UV symmetry $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$ is manifested in the projective algebra between fermion parity and the Majorana translation operator [Rahmani-Zhu-Franz-Affleck '15, Hsieh-Halasz-Grover '16].
- To see this, note $(-1)^F = i^L \chi_1 \chi_2 \cdots \chi_{2L}$.

$$T^{Maj}(-1)^F \left(T^{Maj}\right)^{-1} = i^L \chi_{2L} \chi_1 \cdots \chi_1 = -(-1)^F$$

# Non-invertible symmetry

# Kitaev chain vs. Ising model

- Kitaev chain

$$H = -i \sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$$

- Global symmetries:

$$\mathbb{Z}_2 : \ (-1)^F \chi_\ell (-1)^F = -\chi_\ell$$

$$\mathbb{Z}_{2L} : \ T^{Maj} \chi_\ell \left(T^{Maj}\right)^{-1} = \chi_{\ell+1}$$

- Ising model

$$H_{Ising} = -\sum_{j=1}^{L} X_j - \sum_{j=1}^{L} Z_j Z_{j+1}$$

- Global symmetries:

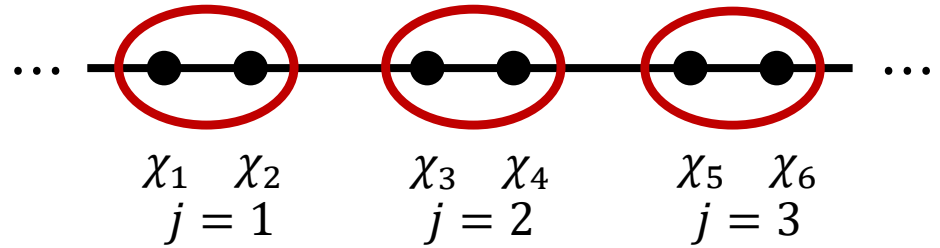
$$\mathbb{Z}_2 : \prod_{j=1}^{L} X_j$$

There is **no** $\mathbb{Z}_{2L}$ symmetry?!

Kitaev chain ≠ Ising model

# Ising model = Kitaev chain + $\mathbb{Z}_2$ gauge field

Gauging $(-1)^F = \prod_{j=1}^{L} i\chi_{2j-1}\chi_{2j}$




1. Introduce $\mathbb{Z}_2$ gauge fields $X_j, Z_j$ on the link $(j-1, j)$.
2. Impose Gauss laws at every site $j$:

$$G_j = Z_j\left(i\chi_{2j-1}\chi_{2j}\right)Z_{j+1} = 1$$

3. Minimally couple the Hamiltonian so that it commutes with Gauss laws:

$$H_{mc} = -i\mu\sum_j \chi_{2j-1}\chi_{2j} - i\mu^{-1}\sum_j \chi_{2j-2}X_j\chi_{2j-1}$$

4. Unitary transformation:

$\chi_{2j-1} \to Z_j\chi_{2j-1}$, $\chi_{2j} \to Z_{j+1}\chi_{2j}$, $X_j \to i\chi_{2j-2}X_j\chi_{2j-1}$, $Z_j \to Z_j$

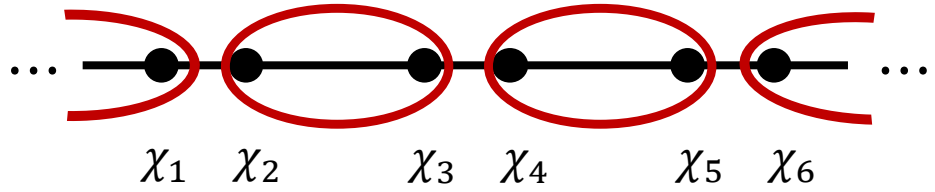
# Ising model = Kitaev chain + $\mathbb{Z}_2$ gauge field

- After these steps, the Gauss law becomes just $G_j = i\chi_{2j-1}\chi_{2j} = 1$, and the Hamiltonian becomes

$$H = -\mu \sum_j Z_{j-1} Z_j - \mu^{-1} \sum_j X_j$$

- The Gauss law eliminates the fermions, and we are left with the Ising model.
- If we had decomposed the fermion parity operator differently, we will find the dual Ising model with $\mu \to \mu^{-1}$.

$$(-1)^F = \prod_{j=1}^{L} i\chi_{2j}\chi_{2j+1}$$



# Majorana translation

- The Majorana translation operator for $\mathbb{Z}_{2L}$ is fermionic:

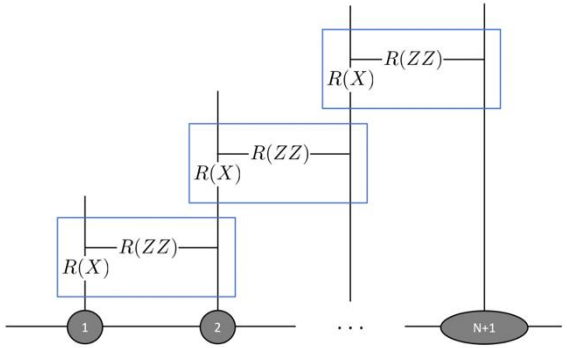
$$T^{Maj}(-1)^F\left(T^{Maj}\right)^{-1} = -(-1)^F$$

$$T^{Maj} = e^{\frac{2\pi i(L-1)}{8}}\chi_L\frac{1+\chi_{2L}\chi_1}{\sqrt{2}}\frac{1+\chi_1\chi_2}{\sqrt{2}}\cdots\frac{1+\chi_{2L}\chi_1}{\sqrt{2}}$$

- It is **not** gauge-invariant in the Ising model.

- Consequence of the anomaly for $\mathbb{Z}_2 \times \mathbb{Z}_{2L}$.

# Kramers-Wannier symmetry




- In fact, the Majorana translation operator becomes the non-invertible Kramers-Wannier symmetry in the Ising model.

$$D = e^{2\pi i \frac{L}{8}} \left( \prod_{j=1}^{L-1} \frac{1 - i\, X_j}{\sqrt{2}} \frac{1 - i\, Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 - i\, X_L}{\sqrt{2}} \frac{1 + \prod_j X_j}{\sqrt{2}}$$

sequential quantum circuit $U_{KW}$ projection

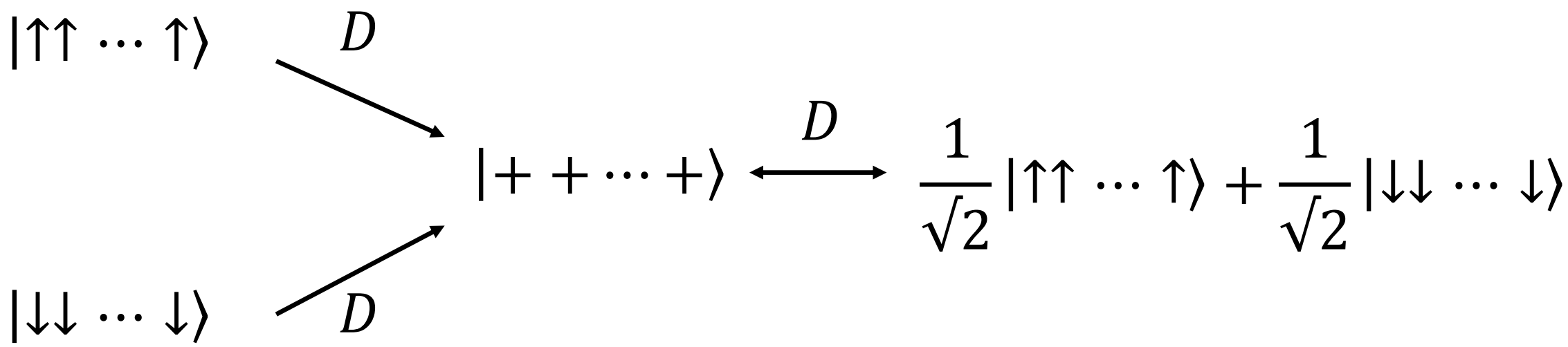
- It implements KW transformation for all sites:

$$D X_j = Z_j Z_{j+1} D, \qquad D Z_j Z_{j+1} = X_{j+1} D$$

- It therefore commutes with the critical Ising model.
- But $D$ does not have an inverse --- non-invertible.

# Non-invertible Kramers-Wannier Symmetry

$$D = e^{2\pi i \frac{L}{8}} \left( \prod_{j=1}^{L-1} \frac{1 - i\, X_j}{\sqrt{2}} \frac{1 - i\, Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 - i\, X_L}{\sqrt{2}} \frac{1 + \prod_j X_j}{\sqrt{2}}$$




$Z|\uparrow\rangle = |\uparrow\rangle, \qquad X|+\rangle = |+\rangle$

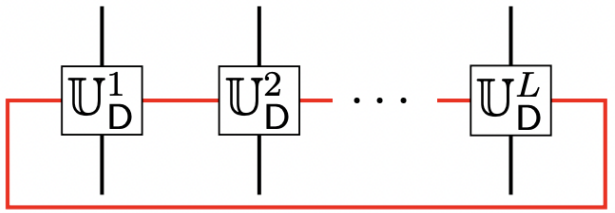
# Matrix Product Operator (MPO)

$$D = e^{2\pi i \frac{L}{8}} \left( \prod_{j=1}^{L-1} \frac{1 - i\, X_j}{\sqrt{2}} \frac{1 - i\, Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 - i\, X_L}{\sqrt{2}} \frac{1 + \prod_j X_j}{\sqrt{2}}$$

- The non-invertible Kramers-Wannier operator commutes with the lattice translation $T$: $O_j \to O_{j+1}$, $DT = TD$.
- It admits a manifestly translationally-invariant presentation as a Matrix Product Operator:

$$D = Tr(\mathbb{U}^1 \mathbb{U}^2 \cdots \mathbb{U}^L)$$

$$\mathbb{U}^j = \begin{pmatrix} |+\rangle\langle 0|_j & |-\rangle\langle 1|_j \\ |-\rangle\langle 0|_j & |-\rangle\langle 1|_j \end{pmatrix}$$



# Non-invertible Kramers-Wannier symmetries on lattices

| | **Tensor product Hilbert space** [Seiberg-SHS '23, Seiberg-Seifnashri-SHS '24, Ma-Li-Cheng '24, Inamura '26, Wen-Inamura-SchaferNameki '26] | **Anyonic chain** [..., Aasen-Fendley-Mong '20] |
|---|---|---|
| Hilbert space | $\otimes_{j=1}^{N}\mathcal{H}_j$ | $\mathcal{H}_{sites} \oplus \mathcal{H}_{links}$ |
| Operator algebra | $D^2 = (1+\eta)T$ mixing with lattice translation | $\mathcal{D}^2 = 1+\eta$ internal |

- It is **not** a group. Here $\eta = \prod_j X_j$.
- The Kramers-Wannier symmetry mixes with lattice translation on a tensor product Hilbert space. This mixing is unavoidable and is related to the Nielsen-Ninomiya-type theorem for lattice chiral symmetries.
- $D^\dagger = DT^{-1}$ is not self-adjoint; no Frobenius-Schur indicator on a tensor prod lattice.

# Lattice and continuum bosonization

| | | |
|---|---|---|
| **Kitaev chain** | | **Ising lattice model** |
| $H = -i\sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$ | gauge $(-1)^F$ → | $H_{Ising} = -\sum_{j=1}^{L} X_j - \sum_{j=1}^{L} Z_j Z_{j+1}$ |
| symmetry $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$<br>$(-1)^F = i^N \chi_1 \chi_2 \cdots \chi_{2N}$ | dual ↔ | symmetry<br>$\eta = X_1 X_2 \cdots X_L$ |
| Maj lattice translation $T^{Maj}$ | ↔ | $D = e^{2\pi i \frac{L}{8}} \left( \prod_{j=1}^{L-1} \frac{1 - i X_j}{\sqrt{2}} \frac{1 - i Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 - i X_L}{\sqrt{2}} \frac{1+\eta}{\sqrt{2}}$<br>$D^2 = (1+\eta)T$ |
| **Continuum Majorana CFT** | | **Continuum Ising CFT** |
| $\mathcal{L} = \frac{i}{2}\chi_L(\partial_t - \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t + \partial_x)\chi_R$ | gauge $(-1)^F$ → | |
| symmetry $G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2^R$<br>$(-1)^F: \chi_L \to -\chi_L, \chi_R \to -\chi_R$ | dual ↔ | symmetry<br>$\eta: \varepsilon \to \varepsilon, \sigma \to -\sigma$ |
| $(-1)^{F_R}: \chi_L \to \chi_L, \chi_R \to -\chi_R$ | ↔<br>$i\chi_L\chi_R = \varepsilon$ | $\mathcal{D}\varepsilon = -\varepsilon\mathcal{D}$<br>$\mathcal{D}^2 = 1 + \eta$ |

# Other topics

- LSM-type constraints: anomalies of non-invertible symmetries. E.g., constraints on RG flows, deconfinement in 1+1d adjoint QCD, …
- Non-invertible symmetries in higher dimensions: non-invertible symmetries in 3+1d QED, 3+1d Wegner duality symmetry, 2+1d examples, …
- Non-invertible SPT phases.
- Applications in particle physics: axions.
- Selection rules for scattering amplitudes.
- …